A model for rapid stochastic distortions of small-scale turbulence

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We present a model describing the evolution of the small-scale Navier–Stokes turbulence due to its stochastic distortion by much larger turbulent scales. This study is motivated by numerical findings (Laval *et al. Phys. Fluids* vol. 13, 2001, p. 1995) that such interactions of separated scales play an important role in turbulence intermittency. We introduce a description of turbulence in terms of the moments of k-space quantities using a method previously developed for the kinematic dynamo problem (Nazarenko *et al. Phys. Rev.* E vol. 68, 2003, 0266311). Working with the k-space moments allows us to introduce new useful measures of intermittency such as the mean polarization and the spectral flatness. Our study of the small-scale two-dimensional turbulence shows that the Fourier moments take their Gaussian values in the energy cascade range whereas the enstrophy cascade is intermittent. In three dimensions, we show that the statistics of turbulence wavepackets deviates from Gaussianity toward dominance of the plane polarizations. Such turbulence is formed by ellipsoids in the k-space centred at its origin and having one large, one neutral and one small axis with the velocity field pointing parallel to the smallest axis.

1. Introduction

Finding a good turbulence model is a long-standing problem. To be useful in applications, the model has to be sufficiently simple and yet capable of capturing the basic physical processes such as the energy cascade and the intermittent bursts. The cascades appear to be a more robust property reasonably well described by classical turbulence closures such as the direct interaction approximation (DIA) (Kraichnan 1961) and its derivatives (e.g. EDQNM, Orszag 1966). The turbulence intermittency appears to be a more subtle process which depends on the detailed features of the dynamical fluid structures. Of particular importance is whether the intermittent bursts are caused by finite-time vorticity 'blow-ups' (believed to become real singularities in the limit of zero viscosity) or a 'slower' exponential vortex stretching by a large-scale strain collectively produced by the surrounding vortex tubes. Note that the first process is local in the scale space – it is usually viewed as two or more vortex tubes of similar and implosively decreasing radius. On the other hand, the second process involves interaction of significantly separated scales – a thin vortex tube and a large-scale strain.

Recent numerical simulations (Laval, Dubrulle & Nazarenko 2001) indicate that it is the non-local scale interactions that are responsible for the deviations of the

structure functions from their Kolmogorov self-similar values whereas the net effect of the local interactions is to reduce these deviations. These conclusions lead to a model of turbulence in which a model closure (e.g. DIA) is used for the local interactions whereas the non-local interactions are described by a wavepacket (WKB) formalism which exploits the scale separation. The latter describes a linear process of distortion of small-scale turbulence by a strain produced by large scales. Such a linear distortion is a familiar process in engineering applications, for example when a turbulent fluid flows through a pipe with a sudden change in diameter. It is described by the rapid distortion theory (RDT) introduced by Batchelor & Proudman (1954). The model considered in this paper is different from the classical RDT in that the distorting strain is stochastic and, therefore, we will call it the stochastic distortion theory (SDT). We will study a simplest version of SDT in which the large-scale strain is modelled by a Gaussian white (in time) noise in the spirit of the Kraichnan model used for turbulent passive scalars (Kraichnan 1974) and of the Kazantsev–Kraichnan model from the turbulent dynamo theory (Kazantsev 1968; Kraichnan & Nagarajan 1967).

Because most studies so far have focused on the theories with local scale interactions, we will devote our attention mainly to the description of the non-local interactions. The local interactions are unimportant at small scales in two dimensions, but they should be taken into account when three-dimensional turbulence is considered. Similarly to RDT, the SDT model deals with k-space quantities because of the much greater simplicity of the pressure term in k-space. We will study statistical moments of the k-space quantities of all orders and not only the second-order correlators as is customary for RDT. The higher k-space correlators carry information about the turbulence statistics and intermittency which is not always available from the two-point coordinate space correlators, the structure functions, which are popular objects in turbulence theory. Indeed, intermittency in some systems can be dominated by singular k-space structures which are not singular in x-space (e.g. periodic fields). These structures leave their signature on the scalings of the k-space moments but not on the x-space structure functions. The system considered in the present paper is of this type, and another example of this kind is the magnetic turbulence in the kinematic dynamo problem (Nazarenko, West & Zaboronski 2003). In fact, SDT has many similarities to the turbulent dynamo problem and in this paper we will use a method developed in Nazarenko *et al.* (2003) for its derivation. We will also see that, like in the dynamo problem, the fourth-order k-space moments allow us to introduce the measures of the mean polarization and of the spectral flatness – quantities of special importance for characterization of the small-scale turbulence.

2. Stochastic distortion of turbulence

Let us consider a velocity field in three-dimensional space that consists of a component U with large characteristic scale L and a component $\boldsymbol{u}$ with small characteristic scale l, $L \gg l$. In this case, Navier–Stokes equation is

$$\partial_t \boldsymbol{U} + \partial_t \boldsymbol{u} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{U} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \nu\nabla^2 \boldsymbol{U} + \nu\nabla^2 \boldsymbol{u}. \quad (2.1)$$

Let us define the Gabor transform (GT) (see Nazarenko 1999; Nazarenko & Laval 2000; Nazarenko, Kevlahan & Dubrulle 2000)

$$\hat{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{k}, t) = \int f(\epsilon^*|\boldsymbol{x} - \boldsymbol{x}_0|) \exp(\mathrm{i}\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{x}_0))\boldsymbol{u}(\boldsymbol{x}_0, t)\, \mathrm{d}\boldsymbol{x}_0, \quad (2.2)$$

where $1 \gg \epsilon^* \gg \epsilon$ and $f(x)$ is a function which decreases rapidly at infinity, e.g. $\exp(-x^2)$. One can think of the GT as a local Fourier transform taken in a box centred at x and having a size which is intermediate between L and l. The GT commutes with the time and space derivatives, ∂_t and ∇. Commutativity with ∂_t is obvious. Note that the GT commutes with ∇ only for distances from the boundaries which are larger than the support of function f. The inverse GT is simply an integration over all wavenumbers, e.g.

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{f(0)} \int \hat{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{k}, t) \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^3}. \tag{2.3}$$

Here, we will study only the non-local interaction of small and large scales and therefore we neglect the nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ which corresponds to local interactions among the small scales. Let us apply the GT to (2.1) with $k \sim 2\pi/l \sim 1 \gg 2\pi/L \sim \epsilon$ and only retain terms up to first power in ϵ and ϵ^* (we chose ϵ^* such that $\epsilon^* \gg \epsilon \gg (\epsilon^*)^2$). All large-scale terms (the first and the third ones on the left-hand side and the third one on the right-hand side) make no contribution because their GT is exponentially small. An equation for the GT of $\boldsymbol{u}$ under such assumptions was obtained in Nazarenko *et al.* (2000):

$$\mathrm{D}_t \hat{\boldsymbol{u}} + (\hat{\boldsymbol{u}} \cdot \nabla)\boldsymbol{U} = \frac{2\boldsymbol{k}}{k^2} \hat{\boldsymbol{u}} \cdot \nabla(\boldsymbol{U} \cdot \boldsymbol{k}) - \nu k^2 \hat{\boldsymbol{u}}, \tag{2.4}$$

where

$$\mathrm{D}_t = \partial_t + \dot{\boldsymbol{x}} \cdot \nabla + \dot{\boldsymbol{k}} \cdot \nabla_{\boldsymbol{k}},$$
$$\dot{\boldsymbol{x}} = \boldsymbol{U}, \tag{2.5}$$
$$\dot{\boldsymbol{k}} = -\nabla(\boldsymbol{k} \cdot \boldsymbol{U}). \tag{2.6}$$

Equation (2.4) provides an RDT description of turbulence generalized to the case when both the mean strain and the turbulence are inhomogeneous. This equation has the form of a WKB-type transport equation with characteristics given by (2.5) and (2.6). Consider this equation for a fluid path determined by $\dot{\boldsymbol{x}}(t) = \boldsymbol{U}$, so that $\hat{\boldsymbol{u}}(\boldsymbol{k}, \boldsymbol{x}, t) \rightarrow \hat{\boldsymbol{u}}(\boldsymbol{k}, \boldsymbol{x}(t), t)$†

$$\partial_t u_m = \sigma_{ij} k_i \, \partial_j u_m - \sigma_{mi} u_i + \frac{2}{k^2} k_m (\sigma_{ij} k_i u_j) - \nu k^2 u_m, \tag{2.7}$$

where $\sigma_{ij} = \nabla_j U_i$ is the strain matrix and operators ∇_i and ∂_i are derivatives with respect to x_i and k_i respectively ($i = 1, \dots, D$). Note that strain σ_{ij} (taken along a fluid path) enters this equation as a given function of time. Equation (2.7) is applicable to arbitrary large-scale flow slowly varying in space. We formulate the SDT model as equation (2.7) complemented by a prescribed statistics of the large-scale flow. One could take, for example, a numerically computed large-scale strain and use it as an input into (2.4) which should later be integrated numerically. In this paper, however, we prefer to derive a reduced model via the statistical averaging which is possible by assuming a sufficiently simple statistics of the large-scale strain. Experiments and numerical data indicate that Navier–Stokes turbulence is Gaussian at large scales and we will use this property in our model. We will further consider a strain that is white

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† Hereafter, we drop hats on $\hat{\boldsymbol{u}}$ because only Gabor components will be considered. Also, we will not mention explicitly the dependence on the fluid path and simply write $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{k}, t)$.

in time as in Nazarenko *et al.* (2003):

$$\sigma_{ij} = \Omega \left(A_{ij} - \frac{A_{ll}}{d} \delta_{ij} \right) \tag{2.8}$$

where A_{ij} is a matrix the elements of which are statistically independent and white in time,

$$\langle A_{ij}(t) A_{kl}(0) \rangle = \delta_{ij} \delta_{kl} \delta(t). \tag{2.9}$$

This choice of strain ensure the incompressibility and statistical isotropy and homogeneity of large-scale velocity increments. In this case

$$\langle \sigma_{ij}(t) \sigma_{kl}(0) \rangle = \Omega \left(\delta_{ik} \delta_{jl} - \frac{1}{d} \delta_{ij} \delta_{kl} \right) \delta(t). \tag{2.10}$$

Note that this is not the only way to satisfy incompressibility and isotropy; there are infinitely many ways to choose such statistics, see Appendix A. However, an additional requirement of statistical homogeneity of the large-scale velocity field removes arbitrariness and leads to the standard expression (Chertkov *et al.* 1999; Falkovich, Gawedzki & Vergasolla 2001; Balkovsky & Fouxon 1999).

$$\langle \sigma_{ij}(t) \sigma_{kl}(0) \rangle = \Omega((d+1)\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl})\delta(t). \tag{2.11}$$

In Appendix A we show that any strain statistics satisfying the requirement of homogeneity of increments only (but not homogeneity of the velocity field itself), differs from (2.11) only by a stochastic uniform rotation around the chosen fluid particle. Such rotation drops out from the rotationally invariant correlators considered in this paper and, therefore, any choice of the strain statistics will lead to (up to a time-scale constant) the same final equations. Thus, we choose strain (2.10) because of its advantages for the simplicity of the analytical and numerical computations. The Gaussian white-in-time strain has also been used in MHD dynamo theory (Kazantsev–Kraichnan model: Kazantsev 1968; Kraichnan & Nagarajan 1967) and in the theory of turbulent passive scalars (Kraichnan 1974; see also the review of Falkovich *et al.* 2001). It is a natural starting point because of its simplicity.

Note that for realistic modelling of small-scale turbulence one has to describe a matching to the large-scale range via a low-k forcing or a boundary condition. As we will see later, some properties of the small-scale turbulence turn out to be independent of these effects, e.g. scalings in the two-dimensional case, whereas the three-dimensional case is more sensitive to the boundary conditions. Detailed modelling of the low-k forcing/boundary conditions is beyond the scope of this paper. Below, we will simply consider the forcing-free evolution of a finite-support initial condition (decaying turbulence). We will also consider finite-flux solutions in two dimensions corresponding to the turbulent cascades in forced turbulence.

3. Generating function

Let us consider the following set of one-point correlators of Gabor velocities:

$$\Psi_s^n = \langle |\boldsymbol{u}(\boldsymbol{k})|^{(2n-4s)} |\boldsymbol{u}(\boldsymbol{k})^2|^{2s} \rangle \tag{3.1}$$

with $n = 1, 2, 3, \ldots$ and $s = 0, 1, 2, 3, \ldots$. Such correlators were shown in Nazarenko *et al.* (2003) to be a fundamental set for homogeneous isotropic turbulence from which one can express any two-point correlators of the following kind:

$$\langle u_{i_1}(\boldsymbol{k}_1) u_{i_2}(\boldsymbol{k}_1) \ldots u_{i_n}(\boldsymbol{k}_1) u_{j_1}(\boldsymbol{k}_2) u_{j_2}(\boldsymbol{k}_2) \ldots u_{j_m}(\boldsymbol{k}_2) \rangle, \tag{3.2}$$

where $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_m$ take values 1, 2 or 3 indexing the components in three-dimensional space, and n and m are arbitrary natural numbers. Note that homogeneity and isotropy in SDT follow from the coordinate independence and isotropy of the strain and it has to be understood only in a local sense, near the considered fluid particle of the large-scale flow.

Now we define a generating function,

$$Z(\lambda, \alpha, \beta, k) = \langle \exp(\lambda |\boldsymbol{u}(\boldsymbol{k})|^2 + \alpha \boldsymbol{u}(\boldsymbol{k})^2 + \beta \overline{\boldsymbol{u}}(\boldsymbol{k})^2) \rangle, \tag{3.3}$$

where an overline denotes complex conjugation. This function allows one to obtain any of the fundamental one-point correlators (3.1) via differentiation with respect to λ, α and β,

$$\Psi_s^n = \left[\partial_\lambda^{(2n-4s)} \partial_\alpha^s \partial_\beta^s Z \right]_{\lambda=\alpha=\beta=0}. \tag{3.4}$$

To derive an evolution equation for Z we will use the methods of functional integration (Furutsu 1963; Novikov 1965) and response functions (see e.g. Schekochihin & Kulsrud 2002). We will adapt this technique to the Fourier space objects in a manner very similar to that used for the turbulent dynamo problem in Nazarenko *et al.* (2003). Let us time differentiate the expression for Z (3.3) and use the dynamical equation (2.7); we have

$$\dot{Z} = k_i \partial_j \langle \sigma_{ij} E \rangle - \lambda \langle \sigma_{ml} (\overline{u}_m u_l + \overline{u}_l u_m) E \rangle - 2\alpha \langle \sigma_{ml} u_m u_l E \rangle - 2\beta \langle \sigma_{ml} \overline{u}_m \overline{u}_l E \rangle$$
$$- 2\nu k^2 \langle (\lambda |\boldsymbol{u}(\boldsymbol{k})|^2 + \alpha \boldsymbol{u}(\boldsymbol{k})^2 + \beta \overline{\boldsymbol{u}}(\boldsymbol{k})^2) E \rangle \quad \text{(3.5)}$$

where

$$E = \exp(\lambda |\boldsymbol{u}(\boldsymbol{k})|^2 + \alpha \boldsymbol{u}(\boldsymbol{k})^2 + \beta \overline{\boldsymbol{u}}(\boldsymbol{k})^2). \tag{3.6}$$

To find the correlators on the right-hand side of (3.5), we use Gaussianity of σ_{ij} and perform a Gaussian integration by parts. Then, we use the whiteness of the strain field to find the response function (functional derivative of u_l with respect to σ_{ij}). Finally, we use the isotropy of the strain so that the final equation involves only $k = |\boldsymbol{k}|$ and no angular coordinates of the wave vector. The derivation is given in Appendix B, we write here only the final result,

$$\dot{Z} = \Omega \left[\left(1 - \frac{1}{d} \right) k^2 Z_{kk} + \frac{1}{d} (4\mathscr{D} + d^2 - 1) k Z_k + 2 \left(1 - \frac{2}{d} + d \right) \mathscr{D} Z - \frac{4}{d} \mathscr{D}^2 Z \right.$$
$$\left. + 2(\lambda^2 + 4\alpha\beta) Z_{\lambda\lambda} + 2\lambda^2 Z_{\alpha\beta} + 8\lambda\alpha Z_{\alpha\lambda} + 8\lambda\beta Z_{\beta\lambda} + 4\alpha^2 Z_{\alpha\alpha} + 4\beta^2 Z_{\beta\beta} \right] - 2\nu k^2 \mathscr{D} Z, \tag{3.7}$$

where the k, α, β and λ subscripts on Z denote differentiation with respect k, α, β and λ respectively and

$$\mathscr{D} = \lambda \partial_\lambda + \alpha \partial_\alpha + \beta \partial_\beta. \tag{3.8}$$

The number of independent variables in this equation can be reduced by taking into account that due to turbulence homogeneity Z depends on α and β only via the combination $\eta = \alpha\beta$ (Nazarenko *et al.* 2003). We then have

$$\dot{Z} = \Omega\left[\left(1 - \frac{1}{d}\right)k^2 Z_{kk} + \frac{1}{d}(4\mathscr{D} + d^2 - 1)kZ_k + 2\left(1 - \frac{2}{d} + d\right)\mathscr{D}Z - \frac{4}{d}\mathscr{D}^2 Z\right.$$

$$\left. + 2(\lambda^2 + 4\eta)Z_{\lambda\lambda} + 2\lambda^2(Z_\eta + \eta Z_{\eta\eta}) + 16\lambda\eta Z_{\eta\lambda} + 8\eta^2 Z_{\eta\eta}\right] - 2\nu k^2 \mathscr{D}Z, \quad (3.9)$$

where

$$\mathscr{D} = \lambda\partial_\lambda + 2\eta\partial_\eta. \tag{3.10}$$

Equation (3.9) is the main equation of SDT. The right-hand side of describes interactions of the separated scales only. In practical applications or numerical modelling one has to add to it a suitable model for the local scale interactions. We leave this task for the future and concentrate below on studying the effect of the non-local interactions only.

4. Two-dimensional turbulence

Let us consider first the two-dimensional case. The large-time dynamics of the small-scale turbulence is known to be dominated in two dimensions by the non-local interactions due to the generation of intense large-scale vortices and, therefore, the SDT model (3.9) is relevant even without including a model for the local interactions. Note that equations for the two-dimensional turbulence in which only non-local interactions are left are formally identical to the passive scalar equations. The energy spectra of the non-local two-dimensional turbulence and the passive scalars in the Batchelor regime were studied in Nazarenko & Laval (2000) without making any assumptions on the strain statistics. Here we will study the higher k-space correlators.

The two-dimensional case is simpler than the three-dimensional one in that all correlators Ψ_s^n with $s > 0$ are not independent and can be expressed in terms of Ψ_0^n, which we will call the energy series,

$$\Psi_0^n \equiv E_n(k, t) = \langle|\boldsymbol{u}(\boldsymbol{k})|^{2n}\rangle = \left[\partial_\lambda^n Z\right]_{\lambda=\eta=0}. \tag{4.1}$$

The equations for correlators E_n are to be obtained by differentiating (3.9) n times with respect to λ and taking $\eta = \lambda = 0$ which gives

$$\dot{E}_n = \frac{\Omega}{2}[k^2(E_n)_{kk} + (3 + 4n)k(E_n)_k + 4n(1 + n)E_n] - 2\nu n k^2 E_n \tag{4.2}$$

Of special interest are the cascade-type solutions realized when turbulence is forced. To obtain these solutions we first re-write (4.2) as a continuity equation in k-space; this can be done in two different ways,

$$\partial_t(k^{2n-1}E_n) = \frac{\Omega}{2}[k^{-1}(k^{2n+2}E_n)_k]_k - 2\nu n k^{2n+1}E_n \tag{4.3}$$

and

$$\partial_t(k^{2n+1}E_n) = \frac{\Omega}{2}[k^3(k^{2n}E_n)_k]_k - 2\nu n k^{2n+3}E_n. \tag{4.4}$$

In the absence of dissipation, $\nu = 0$, equations (4.4) and (4.3) describe conservation of the quantities which have spectral densities $\mathscr{E}_n = k^{2n-1}E_n$ and $\mathscr{F}_n = k^{2n+1}E_n$ respectively. For $n = 1$ these quantities are just the energy and the enstrophy†. We

† Recall that we consider non-local turbulence and, although invariance of enstrophy is easily seen from the vorticity conservation along the fluid paths, the energy invariance is not obvious because there can be non-local energy exchanges between turbulence and the large-scale flow. The conservation of energy was proved for initially *isotropic* turbulence in Nazarenko & Laval (2000).

will call invariants $\mathscr{E}_n$ and $\mathscr{F}_n$ the energy and the enstrophy series respectively. The steady-state solutions in the range where viscosity is negligible are

$$E_n = C_1^{(n)} k^{-2n} \tag{4.5}$$

and

$$E_n = C_2^{(n)} k^{-2n-2}, \tag{4.6}$$

where $C_1^{(n)}$ and $C_2^{(n)}$ are arbitrary positive constants. For $n = 1$ these solutions were obtained in Nazarenko & Laval (2000); solution (4.5) corresponds to a constant energy flux (and equipartition of the enstrophy) whereas (4.6) corresponds to an enstrophy cascade (and equipartition of the energy). Because the equations for E_n are linear, any linear combination

$$E_n = C_1^{(n)} k^{-2n} + C_2^{(n)} k^{-2n-2} \tag{4.7}$$

is also a stationary solution (in fact the general one). Note that the flux of invariant $\mathscr{E}_n$ is given by $-\frac{1}{2}\Omega k^{-1}(k^{2n+2}E_n)_k$ and it is always negative for solutions (4.7) whereas the flux of $\mathscr{F}_n$, which is $-\frac{1}{2}\Omega k^3(k^{2n}E_n)_k$, is always positive in the steady state. Thus, in forced turbulence, solutions (4.5) will form on the low-k side of the forcing scale and solutions (4.6) on the high-k side of it, which agrees with the general observation that the energy cascade is inverse and the enstrophy cascade is a direct one.

Let us introduce a spectral flatness,

$$F_n = E_n/E_1^n. \tag{4.8}$$

For Gaussian fields, F_n would be independent of k and, therefore, the k-dependence of F_n provides information about the scale invariance and presence of turbulence intermittency. In particular, for the solutions (4.5) we have $F_n = \text{const}$, indicating that the inverse cascades are not intermittent: turbulence produced by a Gaussian forcing at some scale k_f will remain Gaussian at $k < k_f$. For solutions (4.6) we have $F_n \sim k^{2n-2}$ which indicates broken scale invariance and growing deviation from Gaussianity at small scales. Such a small-scale intermittency in the direct cascades is due to nearly singular k-structures having the shape of strongly elongated ellipses in two-dimensional k-space which are centred at the origin. Each strain realization will produce just one of these (randomly oriented) ellipses from an initial circular (isotropic) distribution.

5. Nonlocal three-dimensional turbulence

Three-dimensional Navier–Stokes turbulence is never likely to be non-local and for its realistic description one should add a model of the local interactions to the right-hand side of equation (3.9). However, it is still interesting to solve equation (3.9) as is in order to study the effect of pure nonlocal interactions. We will see below that such a study will reveal some interesting physics.

In the three-dimensional case, equation (3.9) is

$$\dot{Z} = \frac{2\Omega}{3}[k^2 Z_{kk} + 4k Z_k + (8 + 2k\partial_k)\mathscr{D}Z + (\lambda^2 Z_{\lambda\lambda} + 8\eta^2 Z_{\eta\eta})$$
$$+ (3\lambda^2 - 4\eta)(Z_\eta + \eta Z_{\eta\eta}) + 16\lambda\eta Z_{\lambda\eta} + 12\eta Z_{\lambda\lambda}] - 2\nu k^2 \mathscr{D}Z, \tag{5.1}$$

The standard procedure to obtain equations for the correlators Ψ_s^n is to differentiate (5.1) with respect to λ and η the required number of times and then to set $\lambda = \eta = 0$. In the three-dimensional case, all the correlators Ψ_s^n are independent and at each

order $2n$ we have a system of coupled equations rather than a single equation to solve as was the case in two dimensions. However, a decoupling arises asymptotically at large times as we will see below. We will start by considering the second- and the fourth-order correlators ($n = 1$ and 2).

5.1. *Energy spectrum*

The energy spectrum is

$$E(k, t) \equiv E_1(k, t) = \langle |\boldsymbol{u}(\boldsymbol{k})|^2 \rangle = [\partial_\lambda Z]_{\lambda = \eta = 0}. \tag{5.2}$$

Differentiating (5.1) with respect to λ and taking the result at $\lambda = \eta = 0$ we have

$$\dot{E} = \frac{2\Omega}{3}(k^2 E_{kk} + 6k E_k + 8E) - 2\nu k^2 E. \tag{5.3}$$

This equation is similar to the Kazantsev equation (Kazantsev 1968) describing the evolution of the magnetic energy spectrum in kinematic dynamo theory. Similarly to dynamo theory, here the total energy grows exponentially and therefore, unlike the two-dimensional case, no stationary cascade states are possible. To have a steady state, one has to add a model of local interactions to SDT, which will be done in future publications. However, we will now study equation (5.3) to examine the consequences of interactions of separated scales.

　　Recently, Schekochihin, Boldyrev & Kulsrud (2002a) presented the solution of the Kazantsev equation obtained by the Kontorovich–Lebedev transform and we will use their results to solve (5.2). By substituting

$$E = e^{7\Omega t/6} k^{-5/2} \phi(k/k_d, t), \quad k_d = \sqrt{\Omega/3\nu}, \tag{5.4}$$

one can reduce (5.3) to

$$\frac{3}{2\Omega}\dot{\phi}(p, t) = p^2 \phi_{pp} + p\phi_p - p^2\phi. \tag{5.5}$$

The right-hand side of this equation is just the modified Bessel operator and by using the Kontorovich–Lebedev transform one immediately obtain for $t \gg 1/\Omega$ (Schekochihin *et al.* 2002a; Nazarenko *et al.* 2003)

$$\phi(p, t) = \text{const} \int_0^\infty \mathrm{d}s\, s \sinh(\pi s)\, K_{is}(p) K_{is}(q) \exp(-s^2 2\Omega t/3), \tag{5.6}$$

where K_{is} is the MacDonald function of an imaginary order and the constant is fixed by the initial condition. At scales much greater than the dissipative one, $p \ll 1$, viscosity is not important for $t \ll (\ln q)^2$ ($q \ll 1$ is the mean wavenumber of the initial condition) and solution (5.6) becomes (Schekochihin *et al.* 2002a; Nazarenko *et al.* 2003)

$$\phi = \text{const}\, t^{-1/2} \exp\left(-\frac{3(\ln k/q)^2}{8\Omega t}\right), \tag{5.7}$$

This solution describes a spectrum with an expanding $k^{-5/2}$ scaling range (which means $k^{-1/2}$ for the one-dimensional energy spectrum). At $t \sim (\ln q)^2$ the front of this

scaling range reaches the dissipative scales and for $t \gg (\ln q)^2$ (5.6) gives

$$\phi = \frac{\text{const}}{t^{3/2}} K_0(p). \tag{5.8}$$

Function $K_0(p)$ decays exponentially at large p which corresponds to a viscous cut-off of the spectrum. For $p \ll 1$, $K_0(p) \approx -\ln p$, which means that at large time the scales far larger than the dissipative one are affected by viscosity via a logarithmic correction,

$$E(k) = \frac{\text{const}}{t^{3/2}} \exp(7\Omega t/6) k^{-5/2} \ln(k_d/k). \tag{5.9}$$

6. Fourth-order correlators, turbulence polarization and flatness

There are two independent fourth-order correlators:

$$S(k,t) = \Psi_0^{(2)} = \langle |u(k)|^4 \rangle = [Z_{\lambda\lambda}]_{\lambda=\eta=0}, \quad T(k,t) = \Psi_1^{(2)} = \langle |u^2(k)|^2 \rangle = [Z_\eta]_{\lambda=\eta=0}. \tag{6.1}$$

Differentiating (5.1) twice with respect to λ and taking the result at $\lambda = \eta = 0$ we have

$$\dot{S} = \frac{2\Omega}{3}(k^2 S_{kk} + 8k S_k + 18S + 6T) - 4\nu k^2 S. \tag{6.2}$$

Now, differentiating (3.9) with respect to η and taking the result at $\lambda = \eta = 0$ we obtain

$$\dot{T} = \frac{2\Omega}{3}(k^2 T_{kk} + 8k T_k + 12T + 12S) - 4\nu k^2 T. \tag{6.3}$$

By subtracting (6.3) from (6.2) we obtain a closed equation for $W = S - T$,

$$\dot{W} = \frac{2\Omega}{3}(k^2 W_{kk} + 8k W_k + 6W) - 4\nu k^2 W. \tag{6.4}$$

The physical meaning of W becomes clear if we re-write it as in Nazarenko *et al.* (2003):

$$W = 4 \sum_{j \neq l}^{3} \langle [\text{Im}(u_j \bar{u}_l)]^2 \rangle = 4 \sum_{j \neq l}^{3} \langle |u_j|^2 |u_l|^2 \sin^2(\phi_j - \phi_l) \rangle \geqslant 0, \tag{6.5}$$

where symbol Im denotes the imaginary part and ϕ_j and ϕ_l are the phases of components u_j and u_l respectively. Thus, we see that W contains information not only about the amplitudes but also about the phases of the Fourier modes. In particular, $W \equiv 0$ corresponds to the case where all Fourier components of the magnetic field have plane polarization. If $W \neq 0$ then other polarizations (circular, elliptic) are present. This is the case for example for Gaussian fields when $W = E^2/2 > 0$. On the other hand, the smallness of the phase differences can be overpowered in W by large amplitudes. Therefore, a better measure of the mean polarization would be a normalized W, e.g.

$$P = W/S. \tag{6.6}$$

Defined in this way the mean turbulence polarization is an example of an important physical quantity which can be obtained from the one-point Fourier correlators and is unavailable from the coordinate space (one-point or two-point) correlators.

Equation (6.4) can be solved similarly to the energy spectrum equation (5.3), namely by transforming it into equation (5.5) by a substitution similar to (5.4) and then using the solution (5.6). In the inviscid regime ($p \ll 1$, $t \ll (\ln q)^2$) we have

$$W = W_0 \, t^{-1/2} \exp(-25\Omega t/6) \, k^{-7/2} \exp\left(-\frac{3(\ln k/q)^2}{8\Omega t}\right), \qquad (6.7)$$

where W_0 is a constant which can be found from the initial conditions. We see that W develops a $k^{-7/2}$ scaling range which is cut off at low and high k by exponentially propagating fronts. Within this scaling range, W decays exponentially in time.

Given W, one can find S by using the substitution $S = V + W/3$, which leads to a closed equation for V which can be solved similarly to E and W. This gives for the inviscid regime

$$S = t^{-1/2} \, k^{-7/2} \exp\left(-\frac{3(\ln k/q)^2}{8\Omega t}\right) \left(V_0 \, \exp(47\Omega t/6) + \tfrac{1}{3} W_0 \, \exp(-25\Omega t/6)\right), \quad (6.8)$$

where V_0 is another constant which can be found from the initial conditions. For $t \gg 1$, the second term in the parentheses should be neglected. Then, we have the following solution for the mean turbulence polarization:

$$P = W/S = \frac{W_0}{V_0} E^{-12\Omega t}. \qquad (6.9)$$

As we see, in the inviscid regime the mean polarization tends to a value that is independent of k which exponentially decays in time. This means that all turbulence wavepackets eventually become plane polarized. Recall that such turbulence is very far from being Gaussian, for which the mean polarization remains finite (elliptic and circular polarized modes are present).

In the diffusive regime, $t \gg (\ln q)^2$, we have

$$W(k) = W_0 \, t^{-1/2} \mathrm{e}^{-25\Omega t/6} \, k^{-7/2} \, \ln(k_d/\sqrt{2}k) \qquad (6.10)$$

and

$$S(k) = V_0 \, t^{-1/2} \mathrm{e}^{47\Omega t/6} \, k^{-7/2} \, \ln(k_d/\sqrt{2}k). \qquad (6.11)$$

Thus, P is still given by the formula (6.9), indicating that the mean polarization continues to further decrease in time at the same exponential rate. Thus, by the time the diffusive regime is achieved W can be essentially put equal to zero.

The fact that the polarization becomes plane has a quite simple physical explanation. Indeed, a vorticity wavepacket of arbitrary polarization will be strongly distorted by the stretching which mostly occurs along the dominant eigenvector of the Lagrangian deformation matrix (corresponding to the greatest Lyapunov exponent). Such a stretching makes any initial 'spiral' flat for large time with the dominant field component lying in the plane passing through the stretching and the wavevector directions (and, of course, $\boldsymbol{u}(\boldsymbol{k})$ is perpendicular to $\boldsymbol{k}$).

Another important measure of turbulence intermittency available from the k-space moments is the spectral flatness which can be defined as $F = S/E^2$. Note that $F - 1$ measures the relative intensity of the k-space fluctuations because it is equal to the square of the ratio of the standard deviation to the mean for the energy distribution in k-space (Nazarenko *et al.* 2003). For large time in the inviscid regime

$$F \sim t^{5/2} \mathrm{e}^{+11\Omega t/2} k^{3/2}. \qquad (6.12)$$

We see that the flatness grows both in time and in k which indicates the presence of small-scale intermittency. When it reaches high values, the fluctuations of the k-space energy distribution are much greater than the mean spectrum. Such an intermittency can be attributed to the presence of coherent structures in k-space.

One can also find a solution for the fourth-order correlator S in the dissipative regime. This will be done in the next section together with correlators of all higher orders.

7. Large-time behaviour of higher correlators

The observation at the end of the previous section that there is a dominant field component allows us to predict that for large time $|\boldsymbol{u}|^4 \approx |\boldsymbol{u}^2|^2$ in each realization so that $Z_{\lambda\lambda} \approx Z_{\alpha\beta}$. Therefore, property $Z_{\lambda\lambda} = Z_{\alpha\beta}$, if valid initially, should be preserved by the equation for Z. Indeed, by using equation (5.1), the combination $w = Z_{\lambda\lambda} - Z_{\alpha\beta} = Z_{\lambda\lambda} - Z_\eta - \eta Z_{\eta\eta}$ satisfies a closed homogeneous equation. This means that if $w \equiv 0$ at $t = 0$ then it will remain identically zero for all time. Thus, we can consider a class of solutions of (3.9) (corresponding to large-time asymptotics of the general solution) such that $Z_{\lambda\lambda} = Z_{\alpha\beta}$. Assuming this equality in (3.9) and putting $\eta = 0$ we have

$$\dot{Z} = \frac{2\Omega}{3}\left[k^2 Z_{kk} + 4k Z_k + (8 + 2k\partial_k)\lambda Z_\lambda + 4\lambda^2 Z_{\lambda\lambda}\right] - 2\nu k^2 \lambda Z_\lambda. \tag{7.1}$$

This gives the following equations for the correlators $E_n \equiv \langle|\boldsymbol{u}(\boldsymbol{k})|^{2n}\rangle$, which is the correlator of order $2n$,

$$\dot{E}_n = \frac{2\Omega}{3}[k^2(E_n)_{kk} + (2n+4)k(E_n)_k + 4n(n+1)E_n] - 2\nu k^2 n E_n. \tag{7.2}$$

Note that for $n = 1$ this coincides with the energy spectrum equation (5.3) which we have already solved. Moreover, by the substitution

$$E_n = \exp((3n^2 + n - 9/4)2\Omega t/3)\,k^{(-n-3/2)}\,\phi(k\sqrt{n}/k_d, t) \tag{7.3}$$

one can reduce (7.2) to equation (5.5) independent of n for function ϕ the solution of which we already know. For the inviscid regime ($1/\Omega \ll t \ll (\ln q)^2$) we have

$$E_n = \mathrm{const}(n)\, t^{-1/2} \exp\left((3n^2 + n - 9/4)2\Omega t/3 - \frac{3(\ln k/q)^2}{8\Omega t}\right) k^{(-n-3/2)}, \tag{7.4}$$

and in the diffusive regime ($t \gg (\ln q)^2$)

$$E_n = \frac{\mathrm{const}(n)}{t^{3/2}} \exp((3n^2 + n - 9/4)2\Omega t/3)\, k^{(n-3/2)}\, K_0(\sqrt{n}k/k_d). \tag{7.5}$$

This expression agrees with the results obtained in the previous sections for $n = 1$ and $n = 2$. We see that the main effect of the dissipation is the prefactor change $t^{-1/2} \to t^{-3/2}$ and the $K_0(k/k_d)$ form-factor which corresponds to a log-correction at $k \ll 1$ and an exponential cut-off at $k \sim k_d$. Similar results were obtained for the magnetic field moments in Nazarenko *et al.* (2003). In that case it is the exponential cut-off that causes the change in the exponential growth of the mean magnetic energy (Kazantsev 1968; Kulsrud & Anderson 1992) and in the higher x-space moments of the magnetic field (Chertkov *et al.* 1999).

The scalings in (7.4) and (7.5) with respect to the order n contain important information about the small-scale turbulence. In particular the exponential growth in

time at exponent $\sim n^2$ indicates that the turbulence statistics is likely to be log-normal. An equivalent result for the magnetic fields in the kinematic dynamo problem was obtained in Chertkov *et al.* (1999) and Schekochihin *et al.* (2002*b*). The log-normality arises because the strain is a multiplicative noise for the velocity field which becomes nearly one-dimensional and because the time-integrated strain tends to become a Gaussian process. Formally, this result can also be obtained using the random matrix theory of Furstenberg (1963) (see also Falkovich *et al.* 2001), and a more detailed physical explanation can be found in Nazarenko *et al.* (2003).

The k-dependence of the Fourier correlators is also very important because it gives information about the dominant structures in wavenumber space. Suppose that initially the turbulence is isotropic and concentrated in a ball centred at the origin in wavenumber space. For each realization such a ball will stretch into an ellipsoid with one large, one short and one neutral dimension. One can visualize this ellipsoid as an elongated flat cactus leaf with thorns showing the velocity field direction. One consequence of this picture is that the wavenumber space will be more sparsely covered by the ellipsoids at large k which implies large intermittent fluctuations of the velocity field in k-space. These fluctuations could be quantified by the flatness F which was shown in (6.12) to grow as $k^{3/2}$, a clear indication of the small-scale intermittency. As we discussed before, the value of F measures the relative intensity of energy fluctuations in the k-space, and large F mean that the fluctuations are much greater than the spectrum itself. In MHD, such intermittency due to sparse k-space structures was discussed by Zeldovich *et al.* (1984)†. Note that our zero-polarization result reveals further details of the k-space structures. In general, the velocity field (both real and imaginary parts) must be perpendicular to k (due to incompressibility) but otherwise is not restricted in direction with respect to the structure (cactus leaf). However, for our zero-polarization fields, the velocity field is always directed transversely to the cactus leaf (i.e. all cactus thorns must be straight).

8. Sensitivity of SDT to the strain statistics

In the previous sections, the SDT model was formulated and studied assuming that the large-scale strain is a Gaussian white noise process. This assumption, similarly to the Kazantsev–Kraichnan dynamo model, allowed us to obtain some important analytical solutions for the energy spectrum, the polarization, the flatness and higher-order correlators which capture the turbulence intermittency. In real experimental and in numerical turbulence the strain is generally far from being a Gaussian white noise. Thus, it is interesting to study the sensitivity of our SDT model and its predictions to the strain statistics.

In this section, we will numerically simulate (2.7) for two different types of strain. First, we consider a synthetic Gaussian strain field with a finite correlation time τ which is algorithmically generated as

$$\sigma_{ij}(t + \mathrm{d}t) = (1 - \mathrm{d}t/\tau)\,\sigma_{ij}(t) + \Omega\,\sqrt{2\,\mathrm{d}t/\tau}\,\left(A_{ij}(t) - \tfrac{1}{3}A_{ll}\delta_{ij}\right)$$

† Zeldovich *et al.* (1984) suggested that the structures can be either 'disks' or 'ropes'. They correspond to two positive and one negative or two negative and one positive Lyapunov exponents respectively. However, in the time-reversible statistics which we consider one of the Lyapunov exponents is zero as it was pointed out by Chertkov *et al.* (1999). Thus, we have an intermediate (between the disk and the rope) structure with one neutral direction.

where A_{ij} is the same matrix as in (2.9) and $\mathrm{d}t$ is the time step. The r.m.s. of the different strain components in this numerical experiment ranged from 2.9 to 3.6 and the correlation time was 0.02. Thus, the correlation time was about 16 times less than the characteristic strain distortion time. We considered 512 strain realizations of the synthetic field.

In the second numerical experiment, the strain matrix components were obtained from a 512^3 spectral DNS of the Navier–Stokes equations at Reynolds number $R_\lambda \simeq 200$. The strain time series were recorded along 512 fluid paths. The r.m.s. of the different strain components in this case ranged from 6.6 to 9.3 and the correlation time was approximately 0.08. Thus, the correlation time in this case is of the same order as the inverse strain rate, which is natural for the real Navier-Stokes turbulence where both values are of order of the eddy turnover time at the Kolmogorov scale.

Because of the fairly short correlation time and because of the Gaussianity, the first numerical experiment is closer to the Gaussian white-noise analytical model than the second experiment where the strain is not Gaussian and has long time correlations. In a sense, comparison of results of the first experiment and the analytical solutions may be considered a performance test of the numerical method. On the other hand, comparison between the results of the first and the second experiments allows us to establish their sensitivity to the strain statistics.

To compute (2.7) we used a Runge–Kutta scheme of second order in time with a time step of 20 times less than the correlation time for the synthetic case. For the simulation with strain from DNS, a time step identical to the DNS was used (i.e. 200 times less than the correlation time of strain). In both numerical experiments, for each strain realization we consider a distribution of wavepackets (2048 in the synthetic case and 8192 for the DNS) with initial k chosen randomly in a sphere of radius $|k| \simeq 2$. For each wavenumber k, the two Fourier components of the velocity u_1 and u_2 are chosen randomly such that $u_j = \beta_j \gamma \exp(\mathrm{i}2\pi\alpha_j)$ where α_j, β_j are uniform random numbers in the interval $[0, 1]$ and $\gamma = 2.\mathrm{e} - 04$ is a constant (a function of the total number of particles). The last component u_3 is deduced from $k \cdot u = 0$ to represent the incompressibility. However, the minimum value of $|k_3|$ is limited by the condition that when the calculated u_3 appears to be excessively large that wavepacket is discarded. The viscosity ν was set to 10^{-6} in both numerical experiments.

Figures 1(*a*) and 1(*b*) show the energy spectrum at several different moments of time for the first and the second numerical experiment respectively. In both cases one can see an excellent agreement with the theoretical -2.5 slope for time less than t_d, which is about 4.5 and 0.7 for the first and the second experiments respectively. At $t = t_d$, turbulence reaches the dissipative scale and the spectrum attains a log-corrected shape.

Figures 2(*a*) and 2(*b*) show the time growth of the total energy and the energy spectrum at several fixed wavenumbers for the first and the second numerical experiment respectively. One can see that the growth is approximately exponential as predicted by the theory. It is interesting that the theory also predicts a change to a slower exponential growth of the total energy when time crosses t_d. A similar effect is called the dissipative anomaly in the kinematic dynamo theory (Kazantsev 1968; Chertkov *et al.* 1999). This effect is consistent with figure 2(*a*) which shows that at $t = t_d \approx 4.5$ the slope for the total energy gets smaller and becomes approximately equal to the slope of the individual k-components (as predicted by the theory). For the second experiment the growth is also approximately exponential and the change of slope occurs at $t = t_d \approx 0.7$.
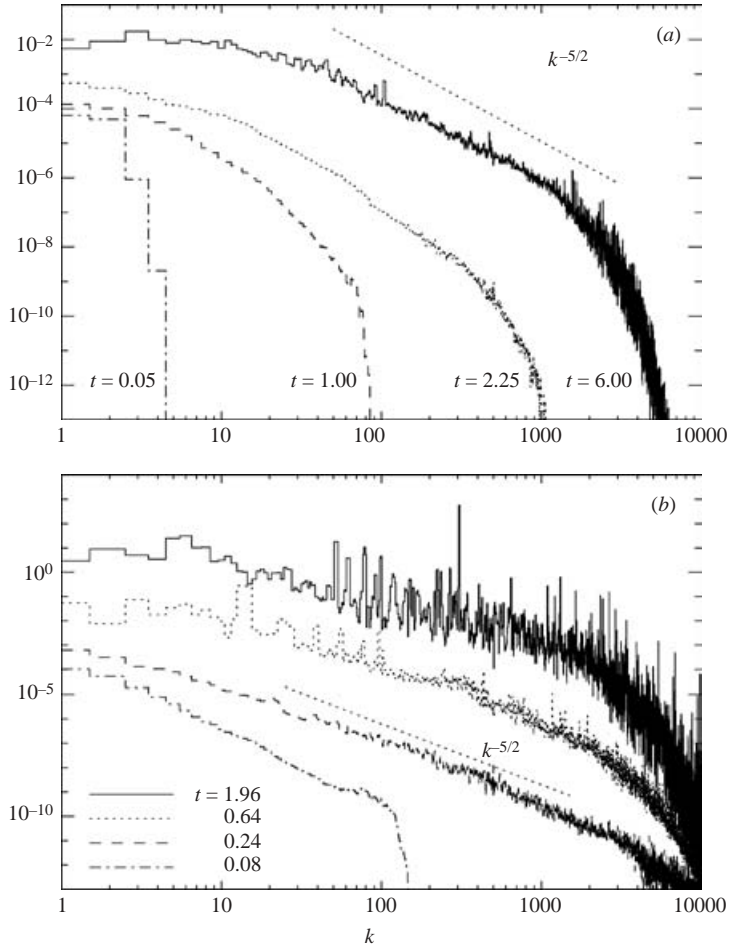
FIGURE 1. Energy spectrum in the case of (*a*) the synthetic Gaussian strain with a finite correlation time and (*b*) the strain obtained from 512^3 DNS.

Figures 3(*a*) and 3(*b*) show the spectrum of the polarization at several fixed wavenumbers for the first and the second numerical experiment respectively. Similarly to the white-noise strain, in both simulations the polarization sharply decreases in time and it tends to an spectrum that is independent of k in the inertial range (with a log-correction for $t > t_d$).

Figures 4(*a*) and 4(*b*) show the spectrum of the flatness at several fixed wavenumbers for the first and the second numerical experiment respectively. Similarly to the white-noise predictions, the flatness is growing with time. However, the theoretical 3/2 slope is observed neither for the synthetic nor for the DNS strain case. The fact that these deviations are observed in the same way for both the DNS strain and the synthetic strain (which is short correlated and therefore quite close to the white-noise process) might indicate a failure of the numerical method to reproduce some features of higher correlators rather than a true deviation due to the differences in the strain statistics. Further discrepancy arises for the DNS strain case at very large time when the flatness is observed to decrease.
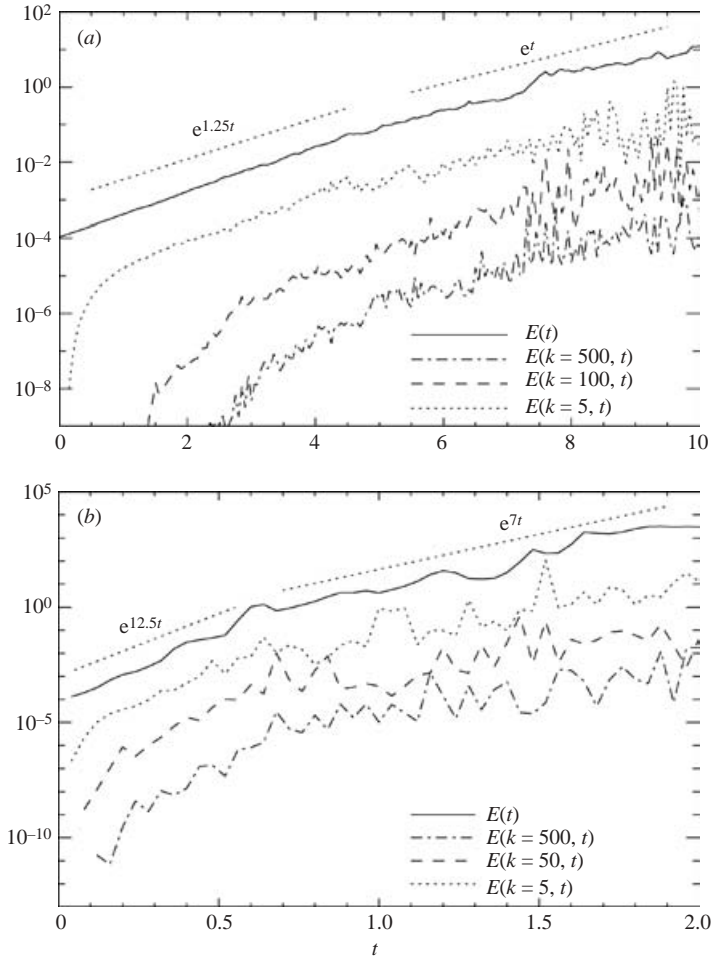
FIGURE 2. Time growth of the total small-scale energy (solid line) and of the energy spectrum at several fixed k (dashed lines) in the case of (a) the synthetic Gaussian strain with a finite correlation time and (b) the strain obtained from 512^3 DNS.

9. Conclusion

In this paper we have introduced a description of the small-scale Navier–Stokes turbulence with much larger scales, the SDT model. Such non-local interactions dominate in two-dimensional turbulence at large time. In three dimensions, they were shown to be responsible for intermittency in numerical experiments (Laval *et al.* 2001). The SDT model assumes that the large scales are Gaussian and short time correlated and it describes turbulence in terms of the one-point k-space correlators using the method introduced in Nazarenko *et al.* (2003) for the kinematic dynamo problem. We studied both two-dimensional and three-dimensional turbulence, the two-dimensional case being equivalent to the problem of two-dimensional passive scalars in the Batchelor regime. In two dimensions, we found steady-state solutions for correlators of all orders. These solutions correspond to forced turbulence and they describe cascades of the energy and enstrophy series of invariants (two invariants at each order). The energy cascades are non-intermittent: initially Gaussian turbulence at the
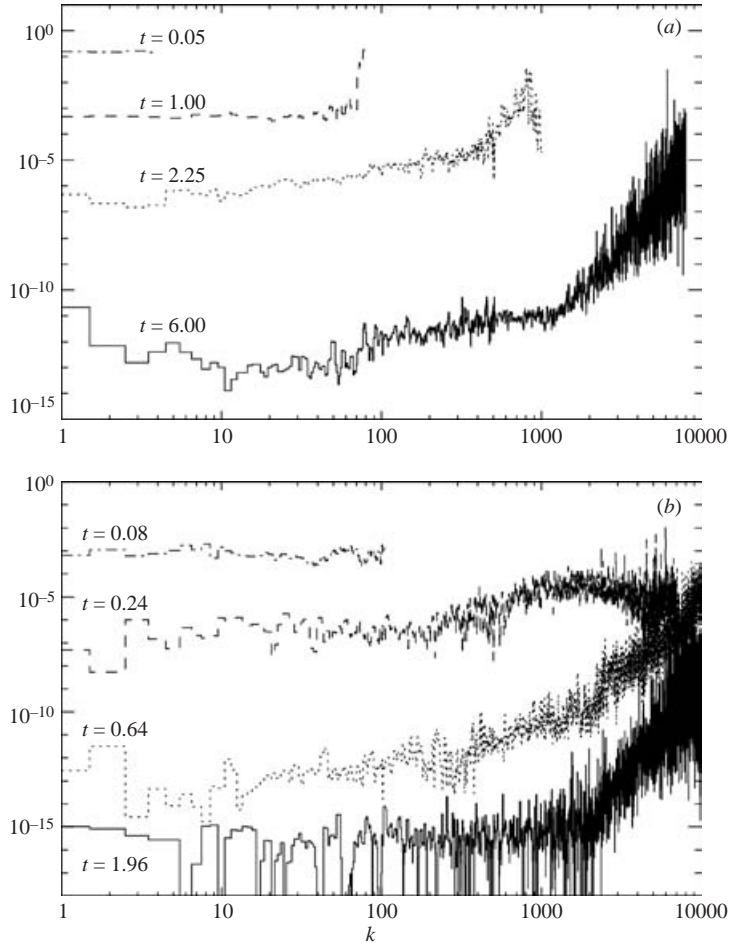
FIGURE 3. Polarization spectrum in the case of (*a*) the synthetic Gaussian strain with a finite correlation time and (*b*) the strain obtained from 512^3 DNS.

forcing scale remains Gaussian and scale invariant throughout the inertial range. On the other hand, the scale invariance and Gaussianity break down for the enstrophy cascades and a regime dominated by thin elliptical structures develops in k-space. It is interesting that similar conclusions about the non-intermittent nature of the inverse cascade were made in experimental work of Paret & Tabeling (1998). In three dimensions, we found that the steady state does not exist in SDT and the total energy grows in a dynamo-like fashion. To have a realistic description of the steady state of the Navier-Stokes turbulence one has to complement SDT with a model for the local scale interactions, which will be done in future work. The simplest way to 'fix' stationarity is to add a linear friction-like dissipation (modelling the effect of the local interactions). However, a more realistic model for local interactions should be nonlinear. One of the possibilities here would be to describe the local interactions by a phenomenological nonlinear k-space diffusion as in the model by Leith (1967). However, even the study of the purely nonlocal interaction presented in this paper reveals several interesting effects which are likely to persist in some form when the local interactions are taken into account. In particular, the non-local
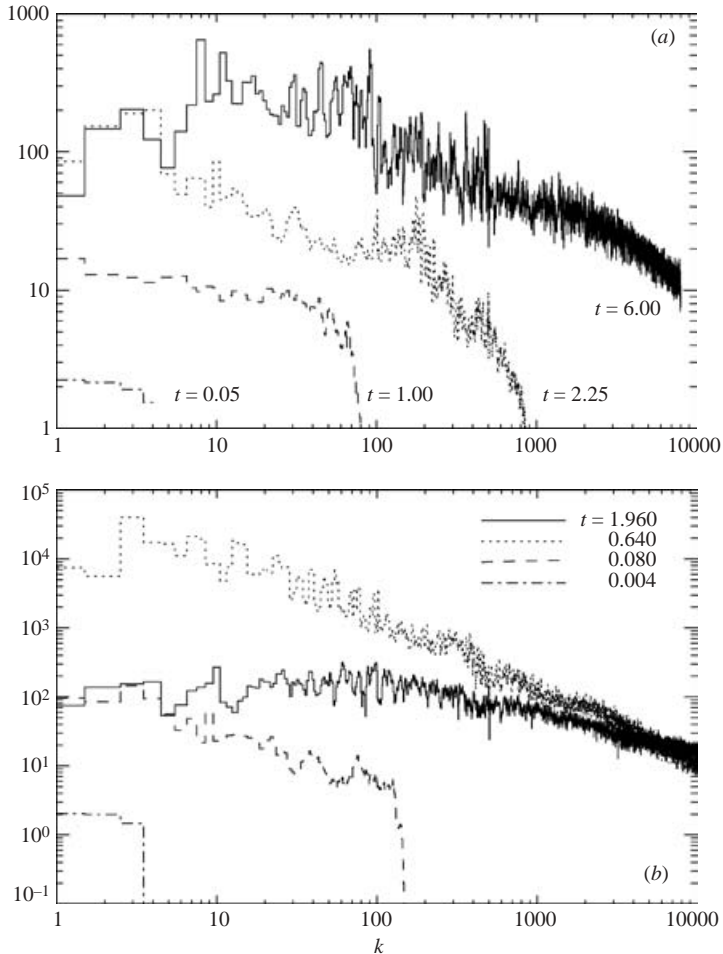
FIGURE 4. Flatness spectrum in the case of (*a*) the synthetic Gaussian strain with a finite correlation time and (*b*) the strain obtained from 512^3 DNS.

interactions are shown to lead to an interesting turbulent state in which all modes have plane polarization in k-space. The statistics of such turbulent fields is far from being Gaussian (in which all polarizations are present). A similar effect for dynamo magnetic fields was found in Nazarenko *et al.* (2003). The k-space moments allow us also to quantify the dominant coherent structures in k-space which are responsible for intermittency in similar way to which the x-space moments capture the x-space structures. Note that singular structures in k-space are not necessarily singular in x-space (e.g. a field periodic in some direction is a singular one-dimensional line in k-space). For Navier–Stokes turbulence, the intermittent k-space structures are found to be very elongated ellipsoids centred at the origin. These structures leave their signatures on the spectral flatness and on the scalings of the higher k-space moments with respect to the order n. It is interesting to establish a relation of these k-space structures with the x-space vorticity 'pancakes' and 'filaments' often discussed in the literature in the vorticity blow-up and intermittency context (see e.g. Frisch 2003). In fact, the elongated k-space ellipse corresponds not to one but to the whole stack of pancakes, i.e. a layered structure discussed in the MHD context by Schekochihin *et al.*

(2002*b*). Moreover, our vanishing polarization result suggests that that the vorticity direction is the same (or strictly opposite) for all of the pancakes in the stack. Further, similarly to the kinematic dynamo problem the scalings indicate the presence of the log-normal statistics of the small-scale velocity fields. Naturally, putting the local interactions back into the model will weaken the tendency to form elongated ellipsoids and plane polarized wavepackets, but some deviation from the Gaussian statistics in the direction predicted by these tendencies should be expected in real Navier–Stokes turbulence.

We studied numerically the SDT model (2.7) for a synthetic Gaussian strain with a finite correlation time and for a strain field obtained from a 512^3 spectral DNS of the Navier-Stokes equation. The results show that most predictions of the white-noise theory, such as e.g. the energy spectrum shape, the polarization decrease and the increase of the spectral flatness, are also observed in these two cases of the strain. Thus, the SDT equations obtained for the Gaussian white noise strain are likely to be a good model for the non-local interactions in real Navier-Stokes turbulence.

Appendix A. Strain statistics

Let $u(x, t)$ be an incompressible smooth random velocity field, $x \in R^d$. Assume that the statistics of *increments* of u is T- and R-invariant. (No assumptions about invariance properties of the velocity field itself is made.)

What is the most general form of the two-point correlation function of the strain matrix, which corresponds to the above velocity field? The requirement of smoothness combined with RT-invariance leads to

$$\langle (u_i(x_1, t) - u_i(x, t))(u_j(x_2, t') - u_j(x, t')) \rangle$$
$$= A(t, t')(x_1 \cdot x_2)\delta_{ij} + B(t, t')x_{1,i}x_{2,j} + C(t, t')x_{1,j}x_{2,i} + O(x^3), \quad (A\,1)$$

where $x_1 = x - x_1$, $x_1 = x - x_1$ and $O(x^3)$ is a shorthand notation for terms of order three and higher in x_1, x_2. Incompressibility $\nabla \cdot u = 0$ implies that

$$A + B + dC = 0. \quad (A\,2)$$

Due to smoothness, $v_i(x_{1(2)}, t) - v_i(x, t) = \sigma_{ij}(t)x_{1(2),\, j} + O(x^2)$. Combining this with (A 1), (A 2) we find the two-point function of strain matrices:

$$\langle \sigma_{ik}(t)\sigma_{jl}(t') \rangle = A(t, t')\left(\delta_{ij}\delta_{kl} - \frac{1}{d}\delta_{jk}\delta_{il} \right) + B(t, t')\left(\delta_{ik}\delta_{jl} - \frac{1}{d}\delta_{jk}\delta_{il} \right). \quad (A\,3)$$

If we assume that the increments of the velocity field u are de-correlated in time and stationary, the above expression will reduce to

$$\langle \sigma_{ik}(t)\sigma_{jl}(t') \rangle = \left(\alpha \left(\delta_{ij}\delta_{kl} - \frac{1}{d}\delta_{jk}\delta_{il} \right) + \beta \left(\delta_{ik}\delta_{jl} - \frac{1}{d}\delta_{jk}\delta_{il} \right) \right) \delta(t - t'), \quad (A\,4)$$

for some constants α and β. Thus, modulo an overall factor, there exists a one-parameter family of two-point correlations of RT-invariant increments of a smooth velocity field.

It is important to stress that the velocity field with increments satisfying (A 1) is not T-invariant in general. Really, R- and T-invariance of u leads, as it is easy to see, to the symmetry of (A 4) with respect to permuting indices i and j. This is true only

if $\alpha = (d+1)\beta$, in which case (A 4) reduces to a standard expression

$$\langle \sigma_{ik}(t)\sigma_{jl}(t') \rangle = \beta((d+1)\delta_{ij}\delta_{kl} - \delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})\delta(t-t'). \tag{A 5}$$

Our choice of strain (2.10) is different from this expression because of its greater simplicity and because it leads (as any strain of the family (A 4)) to the same R-invariant correlators. Indeed, in two and three dimensions a random velocity field with strain statistics satisfying (A 4) can be realized as a linear combination of a T-invariant velocity field and a velocity field corresponding to global random rotation,

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U}(\boldsymbol{x}, t) + \omega(t)(\boldsymbol{n}(t) \times \boldsymbol{x}), \tag{A 6}$$

where $\boldsymbol{U}$ is a random TR-invariant velocity field, $\boldsymbol{n}(t)$ is a random unit vector uniformly distributed on a $(d-1)$-dimensional sphere, $\omega(t)$ is a random angular velocity.

We conclude that correlation functions of physical quantities, which are invariant with respect to global rotations, such as the kinetic energy of fluctuations (and higher correlators considered in this paper), will be the same for the standard model of strain (A 5) and more general strain (A 4). On the other hand, we can use this invariance to choose coefficients in (A 4) to simplify the computations as much as possible, and this choice corresponds to (2.10).

Appendix B. Equation for the generating function

Our aim here is to derive a closed equation for the generating function Z starting with equation (3.5). The last term in this equation is the easiest one,

$$-2\nu k^2 \langle (\lambda |\boldsymbol{u}(\boldsymbol{k})|^2 + \alpha \boldsymbol{u}(\boldsymbol{k})^2 + \beta \overline{\boldsymbol{u}}(\boldsymbol{k})^2)E \rangle = -2\nu k^2 \mathscr{D} Z \tag{B 1}$$

where $\mathscr{D}$ is a differential operator defined in (3.8). The correlators containing factor σ_{ij} can be found using Gaussian integration by parts. In particular

$$\langle \sigma_{ij}E \rangle = \Omega \left\langle \frac{\delta E}{\delta \sigma_{ij}} \right\rangle = \Omega \left[\lambda \langle (\Gamma_{m,ij}\overline{u}_m + \overline{\Gamma}_{m,ij}u_m)E \rangle + 2\alpha \langle \Gamma_{m,ij}u_m E \rangle + 2\beta \langle \overline{\Gamma}_{m,ij}\overline{u}_m E \rangle \right], \tag{B 2}$$

where we have used the definition (3.6). Here, $\Gamma_{m,ij}$ is a response function,

$$\Gamma_{m,ij} = \frac{\delta u_m}{\delta \sigma_{ij}}. \tag{B 3}$$

Differentiating (2.7) with respect to σ_{ij} and using whiteness of the strain tensor we obtain

$$\Gamma_{m,ij} = \left(k_i \, \partial_j + \frac{\delta_{ij}}{d}(1 - k_l \, \partial_l) \right) u_m + \left(\frac{2k_m k_i}{k^2} - \delta_{mi} \right) u_j. \tag{B 4}$$

In what follows we will use the turbulence isotropy, in particular, expressions of the type

$$\langle (\overline{u}_i u_j + \overline{u}_j u_i)E \rangle = \frac{2}{(d-1)} \langle |\boldsymbol{u}|^2 E \rangle \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right), \tag{B 5}$$

$$\langle u_i u_j E \rangle = \frac{1}{(d-1)} \langle \boldsymbol{u}^2 E \rangle \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right), \tag{B 6}$$

$$\langle \overline{u}_i \overline{u}_j E \rangle = \frac{1}{(d-1)} \langle \overline{\boldsymbol{u}}^2 E \rangle \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right). \tag{B 7}$$

Substituting (B 4) into (B 2) and using the above isotropy relations we have

$$\langle \sigma_{ij} E \rangle = \Omega \left(k_i \partial_j - \frac{\delta_{ij}}{d} k_l \partial_l \right) Z + \frac{2\Omega}{(d-1)} \left(\frac{k_i k_j}{k^2} - \frac{\delta_{ij}}{d} \right) \mathscr{D} Z \qquad \text{(B 8)}$$

where $\mathscr{D}$ is a differential operator defined in (3.8). This allows us to find the first term on the right-hand side of (3.5),

$$k_i \partial_j \langle \sigma_{ij} E \rangle = \Omega \left[\frac{(d-1)}{d} k^2 Z_{kk} + \frac{1}{d} (2\mathscr{D} + d^2 - 1) k Z_k + 2 \mathscr{D} Z \right]. \qquad \text{(B 9)}$$

Similarly, the other three terms on the right-hand side of (3.5) can be obtained by Gaussian integration by parts and using the response function (B 4) and the isotropy. After lengthy but straightforward algebra one obtains

$$\lambda \langle \sigma_{ml} (\overline{u}_m u_l + \overline{u}_l u_m) E \rangle = -2\lambda \Omega \left[\left(d - \frac{2}{d} \right) Z_\lambda + 2 \left(1 - \frac{1}{d} \right) \mathscr{D} Z_\lambda \right.$$
$$\left. + \lambda (Z_{\alpha\beta} - Z_{\lambda\lambda}) + \frac{1}{d} k_i \partial_i Z_\lambda \right], \qquad \text{(B 10)}$$

and

$$2\alpha \langle \sigma_{ml} u_m u_l E \rangle = 2\alpha \Omega \left[\left(\frac{2}{d} - d \right) Z_\alpha + \frac{2}{d} \mathscr{D} Z_\alpha - \frac{1}{d} k_i \partial_i Z_\alpha - 2\lambda Z_{\lambda\alpha} - 2\beta Z_{\lambda\lambda} - 2\alpha Z_{\alpha\alpha} \right]. \qquad \text{(B 11)}$$

The fourth term can be obtained from (B 11) via interchanging α with β and $\boldsymbol{u}$ with $\overline{\boldsymbol{u}}$,

$$2\beta \langle \sigma_{ml} \overline{u}_m \overline{u}_l E \rangle = 2\beta \Omega \left[\left(\frac{2}{d} - d \right) Z_\beta + \frac{2}{d} \mathscr{D} Z_\beta - \frac{1}{d} k_i \partial_i Z_\beta - 2\lambda Z_{\lambda\beta} - 2\alpha Z_{\lambda\lambda} - 2\beta Z_{\beta\beta} \right]. \qquad \text{(B 12)}$$

Inserting expressions (B 9), (B 10), (B 11), (B 12) and (B 1), we have the following final equation,

$$\dot{Z} = \Omega \left[\left(1 - \frac{1}{d} \right) k^2 Z_{kk} + \frac{1}{d} (4\mathscr{D} + d^2 - 1) k Z_k + 2 \left(1 - \frac{2}{d} + d \right) \mathscr{D} Z - \frac{4}{d} \mathscr{D}^2 Z \right.$$
$$\left. + 2(\lambda^2 + 4\alpha\beta) Z_{\lambda\lambda} + 2\lambda^2 Z_{\alpha\beta} + 8\lambda\alpha Z_{\alpha\lambda} + 8\lambda\beta Z_{\beta\lambda} + 4\alpha^2 Z_{\alpha\alpha} + 4\beta^2 Z_{\beta\beta} \right] - 2\nu k^2 \mathscr{D} Z. \qquad \text{(B 13)}$$

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